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Conformal invariance and finite one-dimensional quantum chains

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Abstract. Based on previous work of Cardy, we show in a systematic way how using conformal invariance one can determine the anomalous dimensions of various operators from finite quantum chains with different boundary conditions.

The method is illustrated in the case of the three- and four-state Potts models where the anomalous dimensions of the para-fermionic operators are found.

1. Introduction

Recently Belavin *et al* (1984), Dotsenko (1984) and Friedan *et al* (1984) have shown that in two dimensions conformal invariance determines the critical exponents for the bulk correlation functions. Cardy (1984c) has subsequently shown how to use conformal invariance for the surface critical exponents (Binder 1983).

At this point the situation looks as follows. On one hand for a given central charge of the Virasoro algebra (the central extension of the conformal algebra) there is a finite set of operators and their anomalous dimensions. One has to find the physical systems which correspond to the given central charge and how many operators couple to the physical system. On the other hand there are known physical systems (see, e.g. Badke *et al* 1985) where some critical exponents are approximately known but the corresponding central charge of the Virasoro algebra is unknown.

Now in two dimensions conformal invariance has implications on the finite-size scaling (Barber 1983) behaviour of the correlation functions. In the following these implications will be used in trying to determine the central charges and operators which correspond to certain physical systems and their correlation functions.

It was noticed by Luck (1982), Derrida and de Seze (1982), Nightingale and Blöte (1983) and Privman and Fisher (1984) that for a strip with N lattice spacings and *periodic boundary conditions*, the inverse correlation length κ behaves at the critical point of the infinite system like

$$\kappa = AN^{-1} \tag{1.1}$$

where

$$A = 2\pi x \tag{1.2}$$

and x is the scaling dimension of the operator concerned. It was shown by Cardy

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(1984a) that the abovementioned observation is a consequence of conformal invariance. Cardy (1984b) has also shown that the value of the constant A in (1.1) for other boundary conditions is related to the scaling dimensions of other operators.

There is considerable uncertainty in the identification of the central charge due to the only approximate knowledge of the scaling dimensions obtained from (1.2). If one is able to consider several operators, this uncertainty is greatly reduced, and since finite-size scaling allows one to identify new operators, this method is particularly interesting.

Finite-size scaling can be used very efficiently for quantum chains and the extension of the ideas of finite-size scaling on strips to quantum chains which was developed during the last year (Penson and Kolb 1984, Gehlen *et al* 1984, 1985, Burkhardt and Guim 1985) will be used in the present study.

It is the aim of this paper to show in a comprehensive way how to apply the ideas of finite-size scaling and conformal invariance to quantum chains in order to put in evidence various operators and find their anomalous dimensions. Our method is checked for the n-states Potts model ($n = 2, 3, 4$). Other applications to six- and eight-state models will be published elsewhere (Gehlen and Rittenberg 1985). This paper is organised as follows. In § 2 we review the basic knowledge on the unitary representations of the Virasoro algebra. We also show that for a given central charge we have a set of para-fermionic operators (Kadanoff and Ceva 1971, Fradkin and Kadanoff 1980). This observation will be proven useful for the three- and four-states Potts models. In § 3 we briefly review the implications of conformal invariance on finite-size scaling on strips with different boundary conditions. We add here an observation on the behaviour of the energy–energy correlations which will be heavily used in § 5. In § 4 we formulate the consequences of conformal invariance on the finite-size scaling behaviour of Z_n symmetric finite one-dimensional quantum chains. These systems are defined by a Hamiltonian which in some cases can be obtained taking the *anisotropic limit* of the transfer matrix corresponding to a two-dimensional spin system (Kogut 1979) and one might get the false impression that conformal invariance is lost. In fact one may look upon the Hamiltonian as defining the equation of motion of the relevant operators and search for the implication of conformal invariance at the critical point. Among other things conformal invariance fixes the time scale and thus the normalisation of the Hamiltonian. This aspect is discussed in detail for the Ising case in the appendix. The advantage of considering quantum chains is two-fold. The convergence of the estimates for the critical exponents is better than for the transfer matrix (the strip case) and, for a given symmetry of the problem one can find a self-dual Hamiltonian with a known critical point whereas this possibility might be lost in the transfer matrix case.

In § 5 we discuss our numerical results for the Z_3 and Z_4 case. It turns out that the convergence is excellent in the Z_3 case and fair in the Z_4 case. The anomalous dimensions of the Z_4 para-fermionic operators are determined, to our knowledge for the first time. For completeness the known exact results of the Z_2 case are also mentioned. Our conclusions can be found in § 6.

2. Conformal invariance and critical exponents

We first remind the reader of basic knowledge about the applications of conformal invariance to two-dimensional spin systems. If we consider a system defined in the

(x, y) plane and write

$$Z = x + iy, \qquad \bar{Z} = x - iy \tag{2.1}$$

the two-point function is

$$\langle \phi_A(Z_1, \bar{Z}_1) \phi_{A'}(Z_2, \bar{Z}_2) \rangle = \delta_{AA'} (Z_1 - Z_2)^{-2\Delta_A} (\bar{Z}_1 - \bar{Z}_2)^{-2\bar{\Delta}_A} \tag{2.2}$$

where ϕ_A, $\phi_{A'}$ are given operators and Δ_A, $\bar{\Delta}_A$ are the anomalous dimensions. One denotes

$$x_A = \Delta_A + \bar{\Delta}_A, \qquad s_A = \bar{\Delta}_A - \Delta_A \tag{2.3}$$

where x_A is the scaling dimension appearing in (1.2) and s_A is the spin of the operator ϕ_A. Under a conformal transformation

$$w = w(Z) \tag{2.4}$$

one has of course

$$\phi_A(Z, \bar{Z}) \to (\mathrm{d}w/\mathrm{d}Z)^{\Delta_A} (\mathrm{d}\bar{w}/\mathrm{d}\bar{Z})^{\bar{\Delta}_A} \phi_A(w, \bar{w}). \tag{2.5}$$

To a given physical system corresponds a central charge c of two Virasoro algebras given by the generators L_k and $\bar{L}_k$:

$$\begin{aligned} &[L_k, L_{k'}] = (k - k') L_{k+k'} + \tfrac{1}{12} c (k^3 - k) \delta_{k,-k'} \\ &[L_k, \bar{L}_{k'}] = 0 \end{aligned} \tag{2.6}$$

(the $\bar{L}_k$ generators satisfy the same algebra as L_k).

We are interested here in unitary representations of the algebra (2.6), hence the central charge c in (2.6) is quantised:

$$c = 1 - 6/m(m+1), \qquad m = 3, 4, \ldots \tag{2.7}$$

and so are Δ_A and $\bar{\Delta}_A$:

$$\Delta_{p,q} = \Delta_{m-p, m-q+1} = \frac{[mq - (m+1)p]^2 - 1}{4m(m+1)} \qquad 1 \leqslant p \leqslant m-1,\ 1 \leqslant q \leqslant m \tag{2.8}$$

and a similar expression for $\bar{\Delta}_{p',q'}$.

For the n-states Potts models considered in § 5 one takes m odd in (2.7) and chooses (Dotsenko 1984):

$$n = 4\left(\cos \frac{\pi}{m+1}\right)^2. \tag{2.9}$$

The spinless ($s = 0, \Delta = \bar{\Delta}$) operators are identified as follows. The energy density ε corresponds to

$$\Delta_{2,1} = \frac{x_t}{2} = \frac{m+3}{4m} \tag{2.10}$$

and the order (disorder) operators $\sigma(\mu)$ have the same dimensions and correspond to

$$\Delta_{(m+1)/2,(m+1)/2} = \frac{x_h}{2} = \frac{\eta}{4} = \frac{(m+3)(m-1)}{16m(m+1)} \tag{2.11}$$

Cardy (1984c) has suggested that one can extend the applications of conformal invariance to surface exponents, identifying

$$\Delta_{1,3} = x_{h,s} = \frac{\eta_{\|}}{2} = \frac{m-1}{m+1} \tag{2.12}$$

and that $x_{t,s} = 2$ independent of the system.

For completeness we consider also the n-states tricritical Potts models where one takes m even in (2.7) and chooses

$$n = 4(\cos(\pi/m))^2. \tag{2.13}$$

The identification of the operators is

$$\begin{aligned} \Delta_{1,2} &= \frac{x_t}{2} = \frac{m-2}{4(m+1)} \\ \Delta_{m/2,\, m/2} &= \frac{x_h}{2} = \frac{\eta}{4} = \frac{m^2-4}{16m(m+1)} \end{aligned} \tag{2.14}$$

and one can make an educated guess for the surface exponent:

$$\Delta_{3,1} = x_{h,s} = \frac{\eta_{\|}}{2} = \frac{m+2}{m}. \tag{2.15}$$

As will be shown in §§ 3 and 4 the techniques of finite-size scaling are ideal to test (2.12) and (2.15). Up to now we have considered only spinless operators. It was suggested by Fradkin and Kadanoff (1980) that for a system defined on the Abelian group Z_n one has also operators with spin obtained from the short distance expansion of order and disorder operators and that the corresponding values of the spin should be

$$s = Q\tilde{Q}/n \tag{2.16}$$

where Q, $\tilde{Q} = 1, \ldots, n-1$. Since for the case $n = 2$ one has fermionic operators, for $n > 2$ the corresponding operators are called para-fermions and we will denote them by $\psi_{Q,\tilde{Q}}$. We are able to identify, using (2.8), part of the para-fermionic operators. If one takes m odd, the following operators have spin $s = \Delta - \bar{\Delta} = l/n (l = 1, \ldots, n-1)$:

$$l/n = \Delta_{2l,2l-1} - \bar{\Delta}_{m-2l,m-2l} \qquad n = \tfrac{1}{2}(m+1). \tag{2.17}$$

For m even one has

$$l/n = \Delta_{2l+1,2l} - \bar{\Delta}_{m-2l+1,m-2l+1}, \qquad n = m/2. \tag{2.18}$$

It is amusing to notice that for $n = 2$ one gets $m = 3$ and $m = 4$ (corresponding to the Ising and tricritical Ising model), for $n = 3$ one gets $m = 5$ and $m = 6$ (corresponding to the Z_3 Potts and tricritical Potts models) and presumably $m = 7$ and 8 correspond to systems which contain a Z_4. The four-states Potts model corresponds to $m \to \infty$ (see (2.9) and (2.13)) and has had to be treated with care. One takes the limit with s fixed of (2.17) or (2.18) and gets using (2.8):

$$\Delta - \bar{\Delta} = s \qquad \Delta + \bar{\Delta} = \tfrac{1}{2}(s^2+1). \tag{2.19}$$

For example the operator with $s = \frac{1}{2}$ has $\Delta = \frac{9}{16}$ and $\bar{\Delta} = \frac{1}{16}$.

One can also identify operators with unit spin ($s=1$). One has for $m \geqslant 5$ odd

$$\Delta_{(m+1)/2,(m-3)/2} - \bar{\Delta}_{(m-1)/2,(m-3)/2} = 1 \tag{2.20}$$

and for $m \geqslant 6$ even:

$$\Delta_{m/2+2,m/2} - \bar{\Delta}_{m/2+2,m/2+1} = 1. \tag{2.21}$$

In the m infinity limit one obtains (using (2.8))

$$\Delta = \tfrac{25}{16}, \qquad \bar{\Delta} = \tfrac{9}{16}. \tag{2.22}$$

The values (2.22) are different from those obtained taking $s=1$ in (2.19). We have devoted such a large space to para-fermionic operators because in § 5 we will try to identify them.

3. Conformal invariance and finite-size scaling

It was noticed by Cardy (1984a, b) that at the critical point the two-point correlation function in the z plane determines the correlation function on a strip with *periodic boundary conditions.* Using (2.2) and (2.5) we have

$$\langle \phi_A(w_1, \bar{w}_1)\phi_A(w_2, \bar{w}_2)\rangle$$

$$= \left(\frac{dw_1}{dZ_1}\frac{dw_2}{dZ_2}\right)^{-\Delta_A}\left(\frac{d\bar{w}_1}{d\bar{Z}_1}\frac{d\bar{w}_2}{d\bar{Z}_2}\right)^{-\bar{\Delta}_A}(Z_1-Z_2)^{-2\Delta_A}(\bar{Z}_1-\bar{Z}_2)^{-2\bar{\Delta}_A} \tag{3.1}$$

with

$$w = u + iv = (N/2\pi)\ln Z, \qquad -\infty < u < \infty, -\tfrac{1}{2}N \leqslant v \leqslant \tfrac{1}{2}N. \tag{3.2}$$

For the inverse correlation length κ_A of the two-point function parallel to the strip (v fixed) we get:

$$\lim_{N\to\infty} N\kappa_A = 2\pi x_A. \tag{3.3}$$

By mapping the half-plane into the plane one obtains that the inverse correlation length κ_A for the strip with *free boundary* conditions is related to the surface exponents $\kappa_{A,s}$:

$$\lim_{N\to\infty} N\kappa_A = \pi x_{A,s}. \tag{3.4}$$

Let us now see what happens if one takes other boundary conditions. In order to do so let us consider the partition function of a spin system defined on Z_n (Marcu *et al* 1981) and take a strip geometry:

$$Z = e^{-\mathscr{A}} \tag{3.5}$$

$$\mathscr{A} = \sum_{u=-\infty}^{\infty}\sum_{v=1}^{N}\sum_{k=1}^{n-1} a_k(\omega^{k(\alpha_{u,v}-\alpha_{u+1,v})} + \omega^{k(\alpha_{u,v}-\alpha_{u,v+1})}) \tag{3.6}$$

where

$$0 \leqslant \alpha_{u,v} \leqslant n-1, \qquad \omega = \exp(2\pi i/n) \tag{3.7}$$

and a_k are coupling constants. (For the n-states Potts models they are equal.) One can define $n-1$ types of boundary conditions taking

$$\alpha_{u,N+1}=\alpha_{u,1}+\tilde{Q}, \qquad \tilde{Q}=1,2,\ldots,n-1 \tag{3.8}$$

(the case $\tilde{Q}=0$ corresponds obviously to the periodic case). It was suggested by Cardy (1984b) that the inverse correlation length of an order operator σ_Q for a strip with boundary conditions $\tilde{Q}$ should behave like

$$\lim_{N\to\infty} N\kappa_Q=2\pi(\Delta_{Q,\tilde{Q}}+\bar{\Delta}_{Q,\tilde{Q}}-2\Delta_{\tilde{Q}}). \tag{3.9}$$

Here $\Delta_{\tilde{Q}}=\bar{\Delta}_{\tilde{Q}}$ are the anomalous dimensions of the dual operator $\mu_{\tilde{Q}}$ and $\Delta_{Q,\tilde{Q}}$, $\bar{\Delta}_{Q,\tilde{Q}}$ are the anomalous dimensions of the para-fermionic operator

$$\psi_{Q,\tilde{Q}}\sim\sigma_Q\mu_{\tilde{Q}}. \tag{3.10}$$

Using the method which let Cardy to derive (3.9) it is easy to show that if one considers the energy-energy correlations with boundary conditions $\tilde{Q}$ ($\tilde{Q}\neq 0$) the inverse correlation length is

$$\lim_{N\to\infty} N\kappa_\varepsilon=2\pi \tag{3.11}$$

independent of $\tilde{Q}$. This observation will turn out to be very important when we consider quantum chains.

Finally it was shown by Cardy (1984b) that the interfacial tension Σ at T_c, equal to the difference between the free energies per unit length of a strip with periodic boundary conditions and $\tilde{Q}$ boundary conditions is

$$\lim_{N\to\infty} N\sum=4\pi\Delta_{\tilde{Q}}. \tag{3.12}$$

This equation was verified numerically in the Ising case by Malaspinas (1985).

4. Finite-size scaling for quantum chains

We consider Z_n symmetric quantum chains containing N sites. They are defined by the Hamiltonians

$$H^{(\tilde{Q})}=-\frac{1}{n}\sum_{k=1}^{n-1}\left[a_k\sum_{i=1}^{N}(\sigma_i)^k+\lambda b_k\left(\sum_{i=1}^{N-1}(\Gamma_i)^k(\Gamma_{i+1})^{n-k}+\omega^{(n-k)\tilde{Q}}(\Gamma_N)^k(\Gamma_1)^{n-k}\right)\right]. \tag{4.1}$$

Here the operators σ_i and Γ_i are defined by the tensor products

$$\begin{aligned}\sigma_i&=1\otimes1\otimes\ldots\otimes\underset{\nwarrow\ \text{`}i\text{th' site}}{\sigma}\otimes1\ldots\otimes1\\ \Gamma_i&=1\otimes1\otimes\ldots\otimes\Gamma\otimes1\ldots\otimes1\end{aligned} \tag{4.2}$$

and

$$\sigma=\begin{pmatrix}1 & & & \\ & \omega & & 0\\ 0 & & \ddots & \\ & & & \omega^{n-1}\end{pmatrix}, \qquad \Gamma=\begin{pmatrix}0 & 0 & & \ldots 1\\ 1 & 0 & & \ldots 0\\ & & \ddots & \\ 0 & & 1 & 0\end{pmatrix} \tag{4.3}$$

where $\omega = e^{2\pi i/n}$, a_k and b_k are coupling constants. Hermiticity requires $a_k = a_{n-k}$, $b_k = b_{n-k}$. Self-duality is obtained when $b_k = a_k$. The Potts models are obtained when $a_k = b_k = 1$. The Hamiltonians (4.1) have various symmetries corresponding to different choices of the coupling constants a_k and b_k. In (4.1) $\tilde{Q} = 0, 1, \ldots, n-1$ specifies the boundary conditions. For the case of free boundary conditions one simply drops the last term in (4.1).

The operator

$$\hat{Q} = \sum_{i=1}^{N} q_i (\mathrm{mod}\ n) \tag{4.4}$$

where

$$q = \begin{pmatrix} 0 & & & \\ & 1 & & 0 \\ 0 & & \ddots & \\ & & & n-1 \end{pmatrix} \tag{4.5}$$

commutes with the Hamiltonians and has eigenvalues $Q = 0, 1, \ldots, n-1$. Thus the Hamiltonians $H^{(\tilde{Q})}$ have a block diagonal form corresponding to the different values of Q. We denote those blocks by $H_Q^{(\tilde{Q})}$ and their eigenvalues by $E_Q^{(\tilde{Q})}(r)$ $(r = 0, 1, 2, \ldots)$. $E_Q^{(\tilde{Q})}(0)$ denotes the ground state of $H_Q^{(\tilde{Q})}$, $E_Q^{(\tilde{Q})}(1)$ the first excited state, etc. Notice that translational invariance has to be used in diagonalising $H_Q^{(\tilde{Q})}$ and thus to label the states corresponding to $E_Q^{\tilde{Q}}(r)$. For the case of free boundary conditions the Hamiltonians $H^{(F)}$ will have blocks that will be denoted by $H_Q^{(F)}$.

Assume now that for a given number of sites N, we diagonalise the blocks $H_Q^{(\tilde{Q})}$ and $H_Q^{(F)}$ at the critical point. We can now use the results of the last section and assume that the results valid for the transfer matrix of a spin system (finite-size scaling on a strip) applies to Hamiltonian chains. This extension is not obvious but we think that the example of the Ising chain discussed in detail in the appendix will help the reader in the understanding of this ansatz. From (3.3) we have

$$\lim_{N\to\infty} N(E_Q^{(0)}(0) - E_0^{(0)}(0)) = P_Q^{(0)} = 4\pi\xi\Delta_Q \qquad Q = 1, 2, \ldots, n-1. \tag{4.6}$$

Here $\Delta_Q = \bar{\Delta}_Q$ are the anomalous dimensions of the order operators σ_Q. From (3.12) we have

$$\lim_{N\to\infty} N(E_0^{(\tilde{Q})}(0) - E_0^{(0)}(0)) = P_0^{(\tilde{Q})} = 4\pi\xi\Delta_{\tilde{Q}} \qquad \tilde{Q} = 1, 2, \ldots, n-1 \tag{4.7}$$

where $\Delta_{\tilde{Q}} = \bar{\Delta}_{\tilde{Q}}$ are the anomalous dimensions of the disorder operators $\mu_{\tilde{Q}}$. In (4.6) and (4.7), ξ represents an unknown constant which appears because as opposed to the transfer matrix one can always multiply the Hamiltonians (4.1) by an overall factor. This overall factor can be fixed however if one requires that the equations of motion defined by the Hamiltonians are conformal invariant (see the appendix for the Ising case) or by other requirements derived from conformal invariance as will be shown below. From (3.9) and (3.12) we have

$$\lim_{N\to\infty} N(E_Q^{(\tilde{Q})}(0) - E_0^{(0)}(0)) = P_Q^{(\tilde{Q})} = 2\pi\xi(\Delta_{Q,\tilde{Q}} + \bar{\Delta}_{Q,\tilde{Q}}) \qquad Q, \tilde{Q} = 1, 2, \ldots, n-1. \tag{4.8}$$

Here $\Delta_{Q,\tilde{Q}}$ and $\bar{\Delta}_{Q,\tilde{Q}}$ represent the anomalous dimensions of the para-fermionic operators $\psi_{Q,\tilde{Q}}$.

We now consider the equivalent of the energy-energy correlations. From (3.3) we have

$$\lim_{N\to\infty} N(E_0^{(0)}(1)-E_0^{(0)}(0)) = R^{(0)} = 4\pi\xi\Delta_\varepsilon \tag{4.9}$$

where $\Delta_\varepsilon = \bar{\Delta}_\varepsilon$ are the anomalous dimensions of the energy operator. From (3.11) we get

$$\lim_{N\to\infty} N(E_0^{(\tilde{Q})}(1)-E_0^{(\tilde{Q})}(0)) = R^{(\tilde{Q})} = 2\pi\xi \qquad \tilde{Q}=1,2,\ldots,n-1. \tag{4.10}$$

Similarly we have

$$\lim_{N\to\infty} N(E_Q^{(0)}(1)-E_Q^{(0)}(0)) = R_Q = 2\pi\xi \qquad Q=1,2,\ldots,n-1. \tag{4.11}$$

The last equation has a simple physical meaning. In a conformal invariant system the one-particle spectrum in the Q band verifies the dispersion relation

$$E = |p| \doteq 2\pi K/N, \qquad K=0,1,\ldots. \tag{4.12}$$

Here E is the energy and p the momentum. Thus the energy splitting ΔE between the first excited state ($K=1$) and the ground state ($K=0$) is

$$\Delta E = 2\pi/N \tag{4.13}$$

hence one obtains (4.11) if the Hamiltonian is not properly normalised. Conversely (4.10) and (4.11) can be used to determine ξ.

We now consider the case of free boundary conditions. From (3.4) we have

$$\lim_{N\to\infty} N(E_Q^{(\mathrm{F})}(0)-E_0^{(\mathrm{F})}(0)) = P_Q^{(\mathrm{F})} = \pi\xi x_{Q,s} \qquad Q=1,2,\ldots,n-1 \tag{4.14}$$

where $x_{Q,s}$ is half $\eta_\|$ for the order operator σ_Q. Similarly

$$\lim_{N\to\infty} N(E_0^{(\mathrm{F})}(1)-E_0^{(\mathrm{F})}(0)) = R^{(\mathrm{F})} = 2\pi\xi \tag{4.15}$$

where we have used the fact that $x_{t,s}=2$. Equation (4.15) can also be used to determine ξ but compared with (4.10) and (4.11) the convergence is expected to be worse as usual when one compares free boundary conditions with periodical boundary conditions.

Let us now specialise to the n-states Potts model ($n=2,3$ and 4). These systems have a continuous phase transition at $\lambda=1$. Since these systems are self-dual and have a large symmetry the number of independent matrices $H_Q^{(\tilde{Q})}$ is small. Indeed at $\lambda=1$ we have the symmetry relation

$$H_Q^{(\tilde{Q})} = H_{\tilde{Q}}^{(Q)}. \tag{4.16}$$

We also have for $n=3$

$$H_1^{(1)} = H_2^{(2)} = H_1^{(2)}, \qquad H_0^{(1)} = H_0^{(2)} \qquad H_1^{(\mathrm{F})} = H_2^{(\mathrm{F})}. \tag{4.17}$$

For $n=4$ we have

$$\begin{aligned} &H_1^{(1)} = H_3^{(3)} = H_1^{(3)}, \qquad && H_0^{(1)} = H_0^{(2)} = H_0^{(3)} \\ &H_2^{(1)} = H_2^{(3)}, && H_1^{(F)} = H_2^{(F)} = H_3^{(F)}. \end{aligned} \tag{4.18}$$

From (4.16)-(4.18) follows that (4.6) and (4.7) are the same and are independent of Q. Equations (4.10) and (4.11) also coincide and are Q independent. Finally equation (4.14) is Q independent. In the next section we give the numerical estimates of $P_Q^{(\tilde{Q})}$, $R^{(\tilde{Q})}$, $P_Q^{(F)}$ and R^{F}.

5. Numerical results for the three- and four-states Potts models

We first summarise the exact results for the Ising ($n=2$) chain. One has (Burkhardt and Guim 1985, Gehlen *et al* 1984, 1985):

$$\begin{aligned} &P_1^{(0)}=\tfrac{1}{4}\pi, \qquad P_1^{(1)}=\pi, \qquad P_1^{(F)}=\tfrac{1}{2}\pi \\ &R^{(0)}=R^{(1)}=R^{(F)}=2\pi. \end{aligned} \tag{5.1}$$

We thus obtain

$$\begin{aligned} &\xi=1, \qquad \Delta_Q=\Delta_h=\tfrac{1}{16}, \qquad \Delta_\varepsilon=\tfrac{1}{2}, \qquad \eta_\parallel=1 \\ &\Delta_{1,1}+\bar{\Delta}_{1,1}=\tfrac{1}{2}, \qquad (\Delta_{1,1}=\tfrac{1}{2}, \bar{\Delta}_{1,1}=0) \end{aligned} \tag{5.2}$$

in agreement with (2.10), (2.11), (2.12) and (2.17).

We now consider the three-states Potts model. The values of $R^{(0)}$, $R^{(1)}$ and $R^{(F)}$ for various values of the number N of sites are given in table 1. The estimates for large N obtained using the Van dem Broeck-Schwartz (1979) approximants are given in table 2 (the errors are very subjective). Also in this table are the estimates for $P_1^{(0)}$, $P_1^{(1)}$ and $P_1^{(F)}$ obtained previously (Gehlen *et al* 1984). The various estimates are

Table 1. Values for $R^{(0)}$, $R^{(1)}$ and $R^{(F)}$ for N sites (three-states Potts model).

N	$R^{(0)}$	$R^{(1)}$	$R^{(F)}$
2	4.618 802 154	3.516 611 478	3.829 708 431
3	4.684 658 438	4.546 792 350	4.341 545 971
4	4.665 156 333	4.939 283 234	4.607 832 380
5	4.636 532 131	5.125 531 375	4.768 843 870
6	4.610 183 586	5.227 333 554	4.876 165 211
7	4.587 651 584	5.288 609 093	4.952 683 534
8	4.568 621 751	5.328 145 545	5.009 979 488
9	4.552 490 080	5.355 025 785	5.054 503 072
10	4.538 702 229	5.374 061 384	5.090 118 384
11	4.526 804 332	5.387 988 284	5.119 276 499
12	4.516 442 421	5.398 452 525	5.143 605 300
13	4.507 340 257		
14	4.499 281 621		

Table 2. Estimates and predictions for the R and P quantities (three-states Potts model).

	Estimates	Prediction		Estimates	Prediction
$R^{(0)}$	4.42–4.47	4.35	$P_1^{(0)}$	0.725–0.726	0.725
$R^{(1)}$	5.44	5.44	$P_1^{(1)}$	2.53–2.54	2.54
$R^{(F)}$	5.39–5.45	5.44	$P_1^{(F)}$	1.80–1.82	1.81

compared with the predictions obtained choosing $\xi = 5.44/2\pi$ and taking

$$\begin{aligned} &\Delta_Q = \Delta_h = \tfrac{1}{15}, \qquad \Delta_\varepsilon = \tfrac{2}{5}, \qquad \eta_\parallel = \tfrac{4}{3} \\ &\Delta_{1,1} + \bar{\Delta}_{1,1} = \tfrac{2}{5} + \tfrac{1}{15} = \tfrac{7}{15}, \qquad \Delta_{1,1} - \bar{\Delta}_{1,1} = \tfrac{1}{3} \end{aligned} \tag{5.3}$$

in agreement with the identification of the operators done in § 3. It is interesting to observe that one obtains the $s = \frac{1}{3}$ para-fermion and not the $s = \frac{2}{3}$ para-fermion. The reason why one operator occurs and not the other is not known to the authors.

One lesson one derives from table 2 is that the convergence is excellent for $P_1^{(0)}$, $P_1^{(1)}$ and $P_1^{(F)}$, very good for $R^{(1)}$ (which determines ξ) and poorer for $R^{(0)}$ (which determines Δ_ε) and $R^{(F)}$ (which provides a check for the value of ξ).

We now consider the four-states model. The values of $P_1^{(0)}$, $P_1^{(1)}$, $P_1^{(2)}$, $P_2^{(2)}$ and $P_1^{(F)}$ for various numbers of sites are given in table 3. The corresponding values for $R^{(0)}$, $R^{(1)}$ and $R^{(F)}$ are shown in table 4. The estimates for the same quantities are given in table 5. (The estimate for $R^{(0)}$ is very unstable.) How about the predictions? We determine ξ from $R^{(1)}$ and take $\xi = 4.96/2\pi$. It is amusing to observe that the values of ξ for $n = 2, 3$ and 4 are very well given by the expression

$$\xi = \tfrac{1}{2} n^{1/(n-1)}. \tag{5.4}$$

If (5.4) is an exact relation or has a more profound meaning remains to be seen.

Table 3. Values for the P quantities for N sites (four-states Potts model).

N	$P_1^{(0)}$	$P_1^{(1)}$	$P_1^{(2)}$	$P_2^{(2)}$	$P_1^{(F)}$
2	0.763 932 0225	2.152 240 94	2.585 786 44	2.000 000 00	1.317 837 25
3	0.723 830 3611	2.227 339 65	2.733 694 51	2.054 224 95	1.474 745 33
4	0.707 687 3386	2.268 882 64	2.795 005 86	2.090 697 38	1.570 768 44
5	0.698 867 8283	2.295 648 80	2.827 919 81	2.116 195 04	1.636 808 09
6	0.693 215 1284	2.314 612 56	2.848 473 28	2.135 085 08	1.685 656 18
7	0.689 222 0907	2.328 922 41	2.862 625 23	2.149 739 97	1.723 624 40
8	0.686 212 8846	2.340 211 12	2.873 049 39	2.161 517 46	1.754 213 73
9	0.683 839 0244	2.349 413 83	2.881 112 02	2.171 245 05	1.779 535 66
10	0.681 902 0520	2.357 107 23	2.887 581 00	2.179 455 44	1.800 946 36
11	0.680 280 2272				

Table 4. Values for the $R^{(0)}$, $R^{(1)}$ and $R^{(F)}$ for N sites (four-states Potts model).

N	$R^{(0)}$	$R^{(1)}$	$R^{(F)}$
2	4.000 000 000	3.236 067 977	3.464 101 615
3	3.908 326 913	4.170 111 928	3.925 050 645
4	3.805 861 043	4.524 072 624	4.164 514 937
5	3.724 244 595	4.691 080 380	4.309 184 691
6	3.660 009 502	4.781 786 166	4.405 568 759
7	3.608 408 026	4.835 992 237	4.474 276 412
8	3.565 991 472	4.870 685 679	4.525 725 409
9	3.530 403 272	4.894 060 129	4.565 713 337
10	3.500 019 578	4.910 445 966	4.597 711 066
11	3.473 695 495		

Table 5. Estimates and predictions for the R and P quantities (four-states model).

	Estimate	Prediction		Estimate	Prediction
$R^{(0)}$	3.2–3.4	2.48	$P_1^{(0)}$	0.63–0.64	0.62
$R^{(1)}$	4.94–4.97	4.96	$P_1^{(1)}$	2.45–2.55	2.63
$R^{(F)}$	4.86–4.94	4.96	$P_2^{(1)}$	2.97–3.03	3.10
			$P_2^{(2)}$	2.22–2.27	
			$P_1^{(F)}$	2.2–2.3	2.48

The prediction for $P_1^{(0)}$ was obtained taking $\Delta_h=\frac{1}{16}$ and that for $R^{(0)}$ taking $\Delta_\varepsilon=\frac{1}{4}$. If one uses $R^{(0)}$ in order to get Δ_ε one gets $\Delta_\varepsilon=0.33$.

The prediction for $P_1^{(F)}$ was obtained using Cardy's (1984c) prediction for $\eta_\|$ namely $\eta_\|=2$. Our value for $P_1^{(F)}$ suggests $\eta_\|=1.81$. If one uses $P_1^{(F)}$ and $P_1^{(0)}$ together with $\eta=\frac{1}{4}$ one obtains $\eta_\|=1.77$. Both values are in the middle between the value two and another estimate of $\eta_\|$ obtained using the transfer matrix (Droz *et al* 1985) where one obtains $\eta_\|=1.54$.

We now consider the para-fermionic operators. They are related to $P_1^{(1)}$, $P_2^{(1)}$ and $P_2^{(2)}$. We use (4.8) and (2.19). In agreement with Fradkin and Kadanoff (1980) it is natural to take for $P_1^{(1)}$ the para-fermion with spin $s=\frac{1}{4}$ and for $P_2^{(1)}$ the para-fermion with $s=\frac{1}{2}$. The agreement between the estimate and the prediction is very good indeed. What can we say about $P_2^{(2)}$? If one uses (2.16) one gets $s=1$ which combined with (2.22) would give $P_2^{(2)}=10.5$! We have not been able to find the operator corresponding to $P_2^{(2)}$.

It is interesting to observe that para-fermions with spin $\frac{1}{4}$ and $\frac{1}{2}$ show up but not with $s=\frac{3}{4}$. Like in the Z_3 case we have no explanation for this phenomenon.

6. Conclusions

We have shown (see § 4) how using finite-size scaling for quantum chains with different boundary conditions one can identify various operators and compute their anomalous dimensions. Numerical estimates obtained for the three- and four-states Potts models have allowed us to identify the spin $\frac{1}{3}$ (for Z_3) and spin $\frac{1}{4}$ and $\frac{1}{2}$ (for Z_4) para-fermions. We have also obtained an estimate for the $\eta_\|$ for Z_4 ($\eta_\|\approx 1.8$) in agreement with the prediction $\eta_\|=2$.

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Appendix. The quantum Ising chain and conformal invariance

In this appendix we show how conformal invariance fixes the normalisation of the

Hamiltonian describing a quantum chain. We consider the Z_2 symmetric Hamiltonian

$$H=\sum_i\left[\frac{1+\gamma}{2}\sigma_i^x\sigma_{i+1}^x+\frac{1-\gamma}{2}\sigma_i^y\sigma_{i+1}^y-\frac{h}{2}\sigma_i^z\right] \tag{A1}$$

where σ_i^x, σ_i^y and σ_i^z are Pauli matrices, h and γ $(0<\gamma\leqslant 1)$ are coupling constants. For $\gamma=1$ one recovers the $n=2$ case of (4.1). (We have not written the boundary term explicitly.) This system has an Ising transition for $h=1$ for any non-vanishing γ. Gehlen *et al* (1984, 1985) have shown that for this system one has

$$P_1^{(0)}=\tfrac{1}{2}\pi\gamma, \qquad P_1^{(1)}=\pi\gamma, \qquad P_1^{(\mathrm{F})}=\tfrac{1}{2}\pi\gamma. \tag{A2}$$

Comparing these results with (5.1) we deduce that the normalisation factor ξ of the Hamiltonian (see (4.6), etc) should be equal to γ. In other words, considering the Hamiltonian H/γ one would obtain the same results as from the transfer matrix. We will deduce in two ways that $\xi=\gamma$ directly from (A1) using conformal invariance.

We first ask that the equations of motion are conformal invariant. We perform a Jordan–Wigner transformation (for a review, see, e.g., Kogut 1979) and get

$$H=\tfrac{1}{2}\mathrm{i}\sum_i\left((1+\gamma)A_{i+1}B_i+(1-\gamma)A_iB_{i+1}+2hA_iB_i\right) \tag{A3}$$

where A_i, B_i are Hermitian Clifford matrices:

$$\{A_i, A_j\}=\{B_i, B_j\}=\delta_{ij}, \qquad \{A_i, B_j\}=0. \tag{A4}$$

The (Euclidean) time evolution of the A_j and B_j operators is

$$\begin{aligned} \mathrm{d}A_i/\mathrm{d}\tau&=[H, A_i]=\tfrac{1}{2}\mathrm{i}\gamma(B_{i+1}-B_{i-1})+\mathrm{i}hB_i-\tfrac{1}{2}\mathrm{i}(B_{i+1}+B_{i-1})\\ \mathrm{d}B_i/\mathrm{d}\tau&=[H, B_i]=\tfrac{1}{2}\mathrm{i}\gamma(A_{i+1}-A_{i-1})-\mathrm{i}hA_i+\tfrac{1}{2}\mathrm{i}(A_{i+1}-A_{i-1}). \end{aligned} \tag{A5}$$

We now take the continuum limit, $h=1$ and get (with $B_{j+1}-B_{j-1}\to 2\,\partial B_j/\partial y$)

$$\partial A/\partial\tau=\mathrm{i}\gamma\,\partial B/\partial y, \qquad \partial B/\partial\tau=\mathrm{i}\gamma\,\partial A/\partial y. \tag{A6}$$

Denoting

$$C=(A+B)/\sqrt{2}, \qquad D=(A-B)/\sqrt{2} \tag{A7}$$

and calling $x=\gamma\tau$ (which means that one take as Hamiltonian H/γ) we get

$$\partial C/\partial Z=0, \qquad \partial D/\partial\bar{Z}=0 \tag{A8}$$

with $z=x+\mathrm{i}y$. This shows that H/γ is conformally invariant.

There is a second way to find the normalisation factor. The Hamiltonian (A3) can be diagonalised and one obtains

$$H=f(\gamma, h)+\sum_{K=-(N/2-1)}^{N/2}\left[\left(h-\cos\frac{2\pi K}{N}\right)^2+\gamma^2\sin^2\frac{2\pi K}{N}\right]^{1/2}a_K^+a_K \tag{A9}$$

where a_K are fermionic annihilation operators and f the ground state energy. For $h=1$ and small momenta $p=2\pi K/N$, the one-particle spectrum is

$$E=\gamma|p| \tag{A10}$$

and conformal invariance implies again that $\xi=\gamma$.

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